A FOCUS ON FOCAL SURFACES

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Many classical problems in algebraic geometry have regained interest when techniques from differential geometry were introduced to study them. The modern foundations for this approach has been given by Griffiths and Harris in [6], who obtained in this way several classical and new results in algebraic geometry. More recently, this idea has been successfully followed by McCrory, Shifrin and Varley in [12] and [13] to study differential properties of hypersurfaces in $\mathbb{P}^3$ and $\mathbb{P}^4$. In fact these two papers have greatly influenced the present work.

In this spirit, the subject of this paper is the systematic study of focal surfaces of smooth congruences of lines in $\mathbb{P}^3$. This is indeed a clear example of a topic of differential nature in algebraic geometry. The study of such congruences has been very popular among classical algebraic geometers one century ago. Especially Fano has given many important contributions to this field. An essential ingredient in his work has been the focal surface of the congruence. This point of view has been retaken by modern algebraic geometers, such as Verra and Goldstein, and also by Ciliberto and Sernesi in higher dimension.

What we find amazing in the papers by the classics is how much information they were able to provide about the focal surface of the known examples of congruences, in particular about its singular locus (and more especially about fundamental points). They seemed to have in mind some numerical relations that they never formulated explicitly. And even nowadays such kind of relations would require deep modern techniques, like multiple-point theory, but also this powerful machinery is not a priori enough since some generality conditions need to be satisfied.

As a sample of this, the degree and class of the focal surface –the only invariants easy to compute– can be derived immediately from the Riemann-Hurwitz formula. However these invariants, even in the easiest examples (see Example 2.4 or Remarks after Corollary 4.7) seem to be wrong at a first glance. This is due to the existence of extra components of the focal surface or to the possibility that the focal surface counts with multiplicity, although this was never mentioned explicitly by the classics. Even in [5], these possibilities seem not to have been considered.

The starting point of this work was to understand how the classics predicted the number of fundamental points of a congruence. We only know of one formula in the literature involving this number, which is however wrong (see Example 1.15 and the remark afterwards). So our first goal was to use modern techniques in order to rigorously obtain some of the classical results on the topic. Specifically, by regarding the focal surface of a congruence as a scheme, we reobtain its invariants (degree, class, class of its hyperplane

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section, sectional genus, and degrees of the nodal and cuspidal curves) and give them a precise sense.

We also restrict our attention to congruences of bisecants to a curve, or flexes to a surface (since they are special cases in the work by Goldstein), or bitangents to a surface (since all the lines of a congruence are bitangent to the focal surface). In particular, we prove that no congruence of flexes to a smooth surface is smooth, and that a congruence of bitangents to a smooth surface is smooth if and only if the surface is a quartic not containing any line. Another important reason to study these types of congruences is that their focal surfaces have the unexpected or multiple components mentioned above. We give a precise geometrical description of these components and also conjecture that these congruences are the only ones for which the focal surface has such a behavior.

In order to obtain all the above results, we combine a local differential analysis with global methods from intersection theory. In fact, we consider that many of the techniques we develop are interesting by themselves.

In section §0, we give the basic definitions about congruences and their focal surfaces. In section §1, we obtain the classical invariants of the focal surface. The key new technique in this section is to use the construction given in [2] of varieties parametrizing infinitely close points of a given variety.

In sections §2, §3 and §4 we obtain all the invariants of the congruences given by bisecants to a smooth curve or bitangents or flexes to a smooth surface in $\mathbb{P}^3$. In these sections we again adapt some natural constructions to our setting. For instance the constructions at the beginning of section §3 are clearly influenced by the ones in [12].

Finally, section §5 is devoted to relate the behavior of congruences of bitangents to a smooth surface to the behavior of general congruences. We give there several examples and conjectures of what we expect to happen in general.

§0. Notations and definitions.

We will work over an algebrically closed field of characteristic zero. We will denote by $G(1,3)$ the Grassmann variety of lines in $\mathbb{P}^3$. If $I \subset \mathbb{P}^3 \times G(1,3)$ is the incidence variety of pairs (x, L) such that x is a point of the line L, then any of the projections p_1 and p_2 provides I with a structure of projective bundle. In fact, $I = \mathbb{P}(\Omega_{\mathbb{P}^3}(2))$ (where $\mathbb{P}$ will always mean for us the space of rank-one quotients), and the tautological quotient line bundle is just the pull-back of the hyperplane line bundle on $G(1,3)$ (considered as a smooth quadric in $\mathbb{P}^5$). On the other hand, if we consider the Euler sequence on $\mathbb{P}^3$

$$0 \to \Omega_{\mathbb{P}^3}(1) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

and pull it back to I via p_1 and then push it down to $G(1,3)$ via p_2 we get the universal exact sequence on $G(1,3)$

(0.1) $$0 \to S^* \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_{G(1,3)} \to Q \to 0.$$

Here S and Q are the rank-two universal vector bundles, and I can also be viewed as $\mathbb{P}(Q)$.

Given a point $x \in \mathbb{P}^3$, we define the *alpha-plane* associated to it as the set $\alpha(x) \subset G(1,3)$ of all lines in $\mathbb{P}^3$ passing through it. Similarly, given a plane $\Pi \subset \mathbb{P}^3$, we define the *beta-plane* associated to it as the set $\beta(\Pi)$ of all lines in $\mathbb{P}^3$ contained in Π. If $x \in \Pi$, we will write $\Omega(x, \Pi)$ for the pencil of lines contained in the plane Π and passing through the point x.

By *congruence* we will mean a surface $X \subset G(1,3)$. Any congruence X has a *bidegree* (a, b), where a (called the *order of the congruence*) is the intersection number of X with an alpha-plane, and b (called the *class of the congruence*) is the intersection number with a beta-plane. Equivalently, $a = c_2(Q_{|X})$, and $b = c_2(S_{|X}) = c_1(Q_{|X})^2 - c_2(Q_{|X})$.

A congruence can be regarded (under the Plücker embedding of $G(1,3)$) as a surface contained in a smooth quadric of $\mathbb{P}^5$. In particular, we can define the *sectional genus* of a congruence as the genus of the curve obtained by intersecting the surface with a hyperplane of $\mathbb{P}^5$. We will usually denote it with g.

A line in $\mathbb{P}^3$ can also be viewed as a line in the dual $\mathbb{P}^{3*}$, so that a congruence $X \subset G(1,3)$ induces another congruence $X^* \subset G(1, \mathbb{P}^{3*})$, which we will call the *dual congruence* of X. It is clear that, if X has degree (a, b) then X^* has bidegree (b, a). A congruence and its dual have the same sectional genus (in fact both Plücker embeddings are naturally isomorphic).

If we restrict the above projections p_1 and p_2 to $I_X := p_2^{-1}(X)$ then we get a map $q_X : I_X \to \mathbb{P}^3$ which is generically $a : 1$ and a map $p_X : I_X \to X$. We have the following definitions:

Definitions: A point $x \in \mathbb{P}^3$ is a *fundamental point* of X if $q_X^{-1}(x)$ is not a finite set. Dually, a *fundamental plane* of X is a fundamental point of X^*, i.e. a plane containing

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infinitely many lines of the congruence. The *focal locus* of X is the branch locus (typically a surface) of q_X. The elements of the focal locus are called *focal points* of X. Dually, a *focal plane* of X is a focal point of X^*. Equivalently ([5] Lemma 4.4), a focal point $x \in \mathbb{P}^3$ is characterized by the fact that there exists a line L of the congruence such that the embedded tangent plane of X at the point represented by L meets the alpha plane $\alpha(x)$ in at least a line of $\mathbb{P}^5$ (i.e. a pencil of lines of $\mathbb{P}^3$). This is in fact the definition of focal point given by Goldstein.

If we write H and K respectively for the classes of the hyperplane section and the canonical divisor of X, and h for the class of the hyperplane section of $\mathbb{P}^3$, it is not difficult to see that $c_1(T_{I_X}) = 2h - K - H$, so that the class of the ramification locus of q_X is $2h + K + H$. In particular, we obtain from here the very well-known result that a general line L of a congruence contains two focal points x_1, x_2 (counted with multiplicity) such that (x_1, L) and (x_2, L) lie in the ramification locus of q_X.

Definition: We will call a *focal line* L of a congruence to a line of a congruence such that all of its points are focal. Again from [5] Lemma 4.4, this means that the embedded tangent plane of X at the point represented by L meets in a pencil all the alpha planes $\alpha(x)$ for which $x \in L$. Then, a line L is focal if and only if its embedded tangent plane is a beta-plane.

Let X_0 be the open set of non-focal lines of a congruence X. Then the restriction of the map $p_X^{-1}(X_0) \to X_0$ to the ramification locus of q_X is finite (typically of degree two, but it could happen a priori that any line contains only one focal point counted twice). Hence, the branch locus of this restriction has at most two components.

Definition: We will call the *strict focal surface* of a congruence X to the closure F_0 of the reduced structure of the branch locus of $p_X^{-1}(X_0) \to \mathbb{P}^3$. To distinguish from this, we usually refer to the focal locus F (as a scheme) as the *total focal surface*.

Remarks: 1) We abused the notation in the above definition. First of all, X_0 could be empty. As observed in the definition of focal line, this would imply that the embedded tangent plane of X at any point is a beta-plane. In this case, the congruence itself is a beta-plane (see for instance [5], Corollary 4.5.1). On the other hand, F_0 could be either a point (and then X is an alpha-plane) or a curve, which would mean that X is the congruence of bisecants to that curve. As we will observe later, such a congruence is only smooth when the curve is a twisted cubic or an elliptic quartic.

2) It is not superfluous to take the reduced structure in the above definition. As we will see in section 3, the focal locus can appear with high multiplicity for congruences of bitangents or flexes to a surface in $\mathbf{P}^3$.

3) The total focal surface could have more components different from F_0 when $X \setminus X_0$ is a curve. Such a curve will have the property that its embedded tangent line at each point is contained in $G(1,3)$, so that the corresponding extra components of the focal surface

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will be developable ruled surfaces or cones. The existence of these ruled surfaces seems not to have considered by Goldstein. In fact, the number of components of the focal surface can be bigger than two, as will be shown in Example 5.3.

We end this section of background definitions and results by recalling a classical invariant for surfaces in $\mathbb{P}^3$ that we will use frequently:

Definition: If $\Sigma \subset \mathbb{P}^3$ is a surface, we will write μ_1 for the class of its hyperplane section. It is clear that a surface and its dual have the same invariant μ_1.

§1. Numerical invariants of the focal surface of a smooth congruence.

Along this section, $X \subset G(1,3)$ will be a smooth congruence of lines in $\mathbb{P}^3$, H and K will be the hyperplane and canonical classes respectively, and F will be the total focal surface of X.

In order to better understand the geometry and the numerical invariants of F (in particular μ_1), it is convenient to work in the complete flag variety of points, lines and planes rather than only in the incidence variety of points and lines. We consider then

$$A_X := \{(x, L, \Pi) \in \mathbb{P}^3 \times X \times \mathbb{P}^{3*} \mid x \in L \subset \Pi\}.$$

Let $q_{13} : A_X \to J \subset \mathbb{P}^3 \times \mathbb{P}^{3*}$ and $q_2 : A_X \to X$ be the obvious projections, J being the incidence variety of points and planes. Our goal is to directly obtain the focal variety in J, so that we construct simultaneously its dual. For this purpose, we analyze the ramification locus of q_{13}. First, we observe that the map q_2 factors $A_X \to I_X \xrightarrow{p_X} X$. The second morphism is the restriction of the projective bundle $p_2 : I \to G(1,3)$, so that $I_X \cong \mathbb{P}(Q_{|X})$, and the tautological line bundle is the pullback of the hyperplane section h of $\mathbb{P}^3$. Similarly, the first morphism is a projective bundle and $A_X \cong \mathbb{P}(p_X^* S_{|X})$, and its tautological line bundle is the pullback of the hyperplane section h^* of $\mathbb{P}^{3*}$. From this, it is not difficult to compute the Chern classes of T_{A_X}:

(1.1) $$c_1(T_{A_X}) = 2h + 2h^* - 2H - K$$

$$c_2(T_{A_X}) = h^2 + 4hh^* + h^{*2} - 3hH - 3h^*H - 2hK - 2h^*K + 2H^2 + 2HK + c_2(T_X)$$

On the other hand, the Chern classes of the incidence variety J are:

$$c_1(T_J) = 3h + 3h^*$$

$$c_2(T_J) = 3h^2 + 10hh^* + 3h^{*2}$$

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Hence the class in A_X of the ramification locus R of q_{13} will be, using Porteous formula (R will be a surface, since A_X has dimension four, and J has dimension five),

$$(1.2) \qquad [R] = 2hh^* + hH + h^*H + hK + h^*K + 2H^2 + 2HK + K^2 - c_2(T_X).$$

Then, for a general line L in the congruence, one expects to find two elements (x_1, L, Π_1), (x_2, L, Π_2) in R, and it seems reasonable to think that x_1, x_2 are focal points for X, that Π_1, Π_2 are focal planes and that each Π_i is the tangent plane of the focal surface at x_i (it is a very well-known result that the set of focal planes is the dual of the focal surface). However, the last of the statements is not true, but Π_1 is the tangent plane of the focal surface at x_2 and reciprocally Π_2 is the tangent plane at x_1. Let us check this in local coordinates.

Fix an element (x, L, Π) in R and choose coordinates z_0, z_1, z_2, z_3 in $\mathbb{P}^3$ so that x is the point of coordinates $(1 : 0 : 0 : 0)$, L is the line $z_2 = z_3 = 0$ and Π is the plane $z_3 = 0$. We can take u, v to be a system of parameters of X at L and assume that near L the lines of the congruence are given by the span of the rows of the matrix

$$\begin{pmatrix} 1 & 0 & f & g \\ 0 & 1 & h & k \end{pmatrix}$$

where f, g, h, k are regular functions in a neighborhood of L. We can take then a system of coordinates λ, u, v, μ for A_X near (x, L, Π) to represent the point $x(\lambda, u, v) = (1 : \lambda : f + \lambda h : g + \lambda k)$ inside the above line $L(u, v)$ and the plane $\Pi(u, v, \mu)$ containing them of equation $z_3 + \mu z_2 = (g + \mu f)z_0 + (k + \mu h)z_1$. On the other hand, we can take affine coordinates a_1, a_2, a_3 to represent the points $(1 : a_1 : a_2 : a_3) \in \mathbb{P}^3$ and affine coordinates u_0, u_1, u_2 to represent the plane $z_3 - u_2 z_2 = u_0 z_0 + u_1 z_1$. We could remove one coordinate to work in J, locally defined as $a_3 - u_2 a_2 = u_0 + u_1 a_1$, but we prefer to keep the symmetry. Therefore a local expression for q_{13} is given by

$$(\lambda, u, v, \mu) \mapsto (\lambda, f + \lambda h, g + \lambda k, g + \mu f, k + \mu h, \mu).$$

Its Jacobian matrix is then

$$\begin{pmatrix} 1 & h & k & 0 & 0 & 0 \\ 0 & f_u + \lambda h_u & g_u + \lambda k_u & g_u + \mu f_u & k_u + \mu h_u & 0 \\ 0 & f_v + \lambda h_v & g_v + \lambda k_v & g_v + \mu f_v & k_v + \mu h_v & 0 \\ 0 & 0 & 0 & f & u & 1 \end{pmatrix}$$

We immediately see that this matrix has not maximal rank if and only if the two middle rows are linearly dependent. Since the four columns of this submatrix are linearly dependent, the local equations of R are:

$$(1.3) \qquad \begin{vmatrix} f_u + \lambda h_u & g_u + \lambda k_u \\ f_v + \lambda h_v & g_v + \lambda k_v \end{vmatrix} = 0$$

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(1.4)
$$\begin{vmatrix} g_u + \mu f_u & k_u + \mu h_u \\ g_v + \mu f_v & k_v + \mu h_v \end{vmatrix} = 0$$

(1.5)
$$\begin{vmatrix} f_u + \lambda h_u & k_u + \mu h_u \\ f_v + \lambda h_v & k_v + \mu h_v \end{vmatrix} = 0$$

Equation (1.3) means that the value of λ is so that $x(\lambda, u, v)$ is a focal point in $L(u, v)$, while (1.4) means that $\Pi(u, v, \mu)$ is a focal plane. For a "general" value of u, v there would be two possible values of λ and μ, and (1.5) should be interpreted as a way of assigning to each of the two focal points in the line one of the two focal planes. The key observation is that, substracting (1.3) multiplied by $\begin{vmatrix} h_u & k_u \\ h_v & k_v \end{vmatrix}$ and (1.4) multiplied by $\begin{vmatrix} f_u & h_u \\ f_v & h_v \end{vmatrix}$ one gets (1.5) multiplied by:

(1.6)
$$\begin{vmatrix} f_u + \lambda h_u & k_u - \mu h_u \\ f_v + \lambda h_v & k_v - \mu h_v \end{vmatrix} = 0.$$

This means that (1.6) is the other way of assigning to each focal point a focal plane (and we want to prove that this is the "right" one).

Assume now for simplicity that we chose L containing exactly two focal points x, x' and contained in two focal planes Π, Π'. Then there are two corresponding local expressions λ, λ' in terms of u, v verifying (1.3) and two local expressions μ, μ' verifying (1.4), and such that each of the pairs (λ, μ) and (λ', μ') verify (1.5), while the pairs (λ, μ') and (λ', μ) verify (1.6). In particular, the assignement

$$(u, v) \mapsto (\lambda, f + \lambda h, g + \lambda k)$$

is a local parametrization of the focal surface near x. However, the tangent plane at it is not Π, but Π'. Indeed, let μ'_0 the nonzero solution of (1.4) for $u = v = 0$. Then $\Pi' = \Pi(0, 0, \mu'_0)$ has equation $z_3 + \mu'_0 z_2 = 0$. To check that Π' is tangent we need to show that, substituting the above parametrization in the equation of Π' we do not get linear terms. The substitution becomes $z_3 + \mu'_0 z_2 = g + \lambda k + \mu'_0(f + \lambda h)$, and we need to check that the partial derivatives vanish at $u = v = 0$ (and hence also $\lambda = 0$). These partial derivatives are:

$$g_u(0, 0) + \mu'_0 f_u(0, 0) \text{ and } g_v(0, 0) + \mu'_0 f_v(0, 0)$$

To check this vanishing we first observe that (1.3) for $u = v = 0$ implies that each vanishing implies the other. On the other hand, (1.6) implies that

$$\begin{vmatrix} f_u(0, 0) & h_u(0, 0) \\ f_v(0, 0) & h_v(0, 0) \end{vmatrix} \mu'_0 = \begin{vmatrix} h_u(0, 0) & g_u(0, 0) \\ h_v(0, 0) & g_v(0, 0) \end{vmatrix}.$$

From this it is easy to conclude that Π' is indeed the tangent plane.

We can now use the above calculations to prove the following:

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Proposition 1.7. *Let X be a smooth congruence, let F be its total focal surface and consider $\tilde{F} \subset A_X$ to be the closure of the set of elements (x, L, Π) such that (x, L) is a ramification point of q_X and Π is the tangent plane to F at a smooth point x. Then:*

1) *$\tilde{F}$ is linked to R in the complete intersection of the pullbacks to A_X of the ramification loci of q_X and q_{X^*}. In particular, the cycle class of $\tilde{F}$ in A_X is*

$$[\tilde{F}] = 2hh^* + hH + h^*H + hK + h^*K - H^2 + c_2(T_X).$$

2) *The focal surface F has degree $2a + 2g - 2$ and, if it is reduced, has class $2b + 2g - 2$, $\mu_1 = a + b + 4g - 4 - K_X^2 + 12\chi(\mathcal{O}_X)$, sectional (geometric) genus $9g - 8 - b + K_X^2$ and $\chi(\mathcal{O}_{\tilde{F}}) = 6g - 6 - a - b + K_X^2 + 2\chi(\mathcal{O}_X)$.*

Proof: The fact that $\tilde{F}$ and R are linked is just the geometrical translation of the computations before the statement. In the previous section we proved that the class of the ramification locus of q_X was $2h + H + K$. By duality, the ramification locus of q_{X^*} will be $2h^* + H + K$. Multiplying these two classes and substracting the cycle class of R we complete the proof of 1).

The degree, μ_1 and class of the focal surface are easy to obtain, by just multiplying the cycle class of $\tilde{F}$ respectively by h^2, hh^* and h^{*2} (we are also using the adjunction identity $KH + H^2 = 2g - 2$ and the Noether formula $c_2(T_X) = 12\chi(\mathcal{O}_X) - K^2$). To compute the other invariants we need to know the Hilbert polynomial of $\tilde{F}$. For this purpose, it is not enough to know the cycle class of $\tilde{F}$, but to use the fact that it is obtained by linkage inside a complete intersection M of divisors of classes $2h + H + K$ and $2h^* + H + K$. This fact implies (see [14], Prop. 1.1) that there is an exact sequence

$$0 \rightarrow \mathcal{I}_M \rightarrow \mathcal{I}_R \rightarrow \mathcal{H}om_{\mathcal{O}_{A_X}}(\mathcal{O}_{\tilde{F}}, \mathcal{O}_M) \rightarrow 0.$$

Now the wanted invariants can be directly obtained from the coefficients of the polynomial $\chi(\omega_{\tilde{F}}(Th)) \in \mathbb{Q}[T]$. We will compute it from the above exact sequence. We first observe that, by adjunction and (1.1), $\omega_M \cong \omega_{A_X|M}(2h + 2h^* + 2H + 2K) \cong \mathcal{O}_M(4H + 3K)$, so that

$$\mathcal{H}om_{\mathcal{O}_{A_X}}(\mathcal{O}_{\tilde{F}}, \mathcal{O}_M) \cong \mathcal{H}om_{\mathcal{O}_{A_X}}(\mathcal{O}_{\tilde{F}}, \omega_M)(-4H - 3K) \cong \omega_{\tilde{F}}(-4H - 3K).$$

We then need to compute $\chi(\mathcal{I}_M(Th + 4H + 3K))$, which is very easy since M is a complete intersection. On the other hand, from the construction of R, there is an exact sequence

$$0 \rightarrow T_{A_X} \rightarrow q_{13}^* T_J \rightarrow \mathcal{I}_R(h + h^* + 2H + K) \rightarrow 0$$

from where we can compute $\chi(\mathcal{I}_R(Th + 4H + 3K))$. With the Maple package Schubert one performs the computations and arrives to the wanted result. $\square$

Remarks: 1) The degree and class of the focal surface are very well-known and there are much simpler ways to compute them. In fact, all the other numerical invariants of the focal surface, except μ_1, can be computed by just using the incidence variety point-line. In fact, μ_1 can also be computed by using that it is the class of X considered as a surface in $\mathbb{P}^5$ (see [15]). Then μ_1 is nothing but the degree of $c_2(P^1(\mathcal{O}_X(1)))$, which is easily seen to be the value just computed.

2) The computations previous to the proof of the above proposition show that, for a general line L of a congruence X, there are exactly two pencils $\Omega(x_1, \Pi_1)$ and $\Omega(x_2, \Pi_2)$ (given by the two branch points of q_{13} on L) that are tangent to X at the point represented by L. Hence the embedded tangent plane of X (as a surface in $\mathbb{P}^5$) at L is the one generated by these two pencils. However, the tangent plane at x_1 of the focal surface F is Π_2 and reciprocally.

Proposition 1.8. *Let X be a smooth congruence, and let F be its total focal surface. Assuming that the only one-dimensional singular locus of F consists of a nodal curve D and a cuspidal curve C, then*

$$\deg(D) = 2a^2 - 10a + 4b + 4ag + 2g^2 - 34g + 32 - 4K_X^2 + 12\chi(\mathcal{O}_X)$$

$$\deg(C) = 3a - 3b + 18g - 18 + 3K_X^2 - 12\chi(\mathcal{O}_X).$$

Proof: The underlying idea is quite simple, although it requires a precise construction of some technical complexity. We just want to study when the fibers of the map $q_X : I_X \to \mathbb{P}^3$ contain three infinitely close points (to find the cusps) or two pairs of infinitely close points (to find the nodes). We will consider more generally the projection $\pi : \mathbb{P}^3 \times X \to \mathbb{P}^3$ and apply to it a theory of infinitely close points of its fibers (which will be just infinitely closed points in X). To avoid some technical difficulties, we will reduce to the case of cuspidal points. Note that, since we know from Prop. 1.7 the geometric genus of the hyperplane section of F, the degree of the nodal curve can be computed at once if we know the degree of the cupidal curve (just apply the Plücker formula to a hyperplane section of F).

So we want to find a variety parametrizing sets of three infinitely close points in the fibers of π, to find their the subset $\tilde{C}$ of those who are in fact on X. We will follow the construction of [2]. Clearly, the variety parametrizing pairs of infinitely close points in the fibers of π is nothing but $\mathbb{P}(\Omega_{\mathbb{P}^3 \times X/\mathbb{P}^3}) = \mathbb{P}^3 \times \mathbb{P}(\Omega_X) =: \mathbb{P}^3 \times D_X^1$. Let $f_1 : D^1X \to X$ the structure projection and write L_1 for the tautological line bundle of D^1X. Now the variety parametrizing sets of three infinitely closed points in the fiber of π is given by $\mathbb{P}^3 \times D^2X$, where $D^2X := \mathbb{P}(G)$, G being the rank-two vector bundle on D^1X defined as

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a push-forward in the following commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_{D^1X/X} \otimes L_1 & \to & f_1^* \Omega_X & \to & L_1 & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_{D^1X/X} \otimes L_1 & \to & \Omega_{D^1X} & \to & G & \to & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & \Omega_{D^1X/X} & = & \Omega_{D^1X/X} & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

(1.9)

(see [2] for more details). Let $f_2 : D^2 X \to D^1 X$ denote the structure projection and let L_2 be the tautological line bundle on $D^2 X$. We are now going to try to restrict the above construction to X, having in mind that we are not only looking for infinitely close points whose support is in the fiber of q_X: we need the infinitesimal information defined by these points to be also in the fiber of q_X.

The first step is conceptually easy. Since we want the infinitely close points to be supported on the fiber of q_X, it suffices to restrict the above construction to I_X rather than working on the whole $\mathbb{P}^3 \times X$. Observe that the inclusion $I_X \subset \mathbb{P}^3 \times X$ is induced by projectivizing the quotient of bundles in the restriction to X of the universal sequence (0.1). Hence, I_X is defined in $\mathbb{P}^3 \times X$ as the zero locus of the natural section of $\pi^* S_{|X} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$. In particular, the class of I_X inside $\mathbb{P}^3 \times X$ is given (we will omit to write pullbacks when they are clear) by

(1.10)
$$[I_X] = h^2 + hH + c_2(\pi^* S_{|X}).$$

Keep the same notations for the above construction restricted to I_X and let us see now when an element of $(1 \times f_1)^{-1}(I_X) \subset \mathbb{P}^3 \times D^1 X$ corresponds to a pair of infinitely close points contained as a scheme in the fiber of q_X. Those elements will be characterized by the fact that the universal quotient $(1 \times f_1)^* \Omega_{\mathbb{P}^3 \times X/\mathbb{P}^3} \to L_1$ factors through $(1 \times f_1)^* \Omega_{I_X}$. This means that the composed map $N^*_{I_X/\mathbb{P}^3 \times X} \to (1 \times f_1)^* \Omega_{\mathbb{P}^3 \times X/\mathbb{P}^3} \to L_1$ is zero.

Since I_X was defined as the zero locus of a section of $\pi^* S_{|X} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$, then its normal bundle $N_{I_X/\mathbb{P}^3 \times X}$ is isomorphic to $q_X^* S_{|X} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$. Hence, the wanted subset $X' \subset (1 \times f_1)^{-1}(I_X) \subset \mathbb{P}^3 \times D^1 X$ is defined as the zero locus of a section of $(1 \times f_1)^* (p_X^* S_{|X} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \otimes L_1$, and its class inside $(1 \times f_1)^{-1}(I_X)$ is then:

(1.11)
$$[X'] = c_1(L_1)^2 + 2c_1(L_1)h + c_1(L_1)H + h^2 + hH + c_2(S).$$

We restrict to that subset and again abuse the notation by not changing it after the restriction. Our final step is to identify inside $(1 \times f_2)^{-1}(X') \subset \mathbb{P}^3 \times D^2 X$ the subset X''

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of those infinitely closed points in the fiber of q_X. The apparently new problem is that now D^2X is not the projectivization of a cotangent bundle, but of its quotient G defined in (1.9). However this is not a problem, since the reasoning is exactly as above. Indeed, we have now a universal epimorphism $(1 \times f_2)^*G \to L_2$ on $(1 \times f_2)^{-1}(X')$, and again X'' is the locus for which the natural composition

$$N^*_{X'/(1 \times f_1)^{-1}(I_X)} \to \Omega_{(1 \times f_2)^{-1}(X')} \to (1 \times f_2)^*G \to L_2$$

is zero. Hence X'' is the zero locus of a section of $(1 \times f_2)^*((1 \times f_1)^*(q_X^* S_{|X} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \otimes L_1) \otimes L_2$, and its class in $(1 \times f_2)^{-1}(X')$ is then

(1.12) $$[X''] = c_1(L_2)^2 + 2c_1(L_1)c_1(L_2) + 2c_1(L_2)h + c_1(L_2)H$$

$$+ c_1(L_1)^2 + 2c_1(L_1)h + c_1(L_1)H + h^2 + hH + c_2(S_{|X}).$$

We finally observe that the expressions (1.10), (1.11) and (1.12) can be lifted to classes in $\mathbb{P}^3 \times D^2X$, so that the degree of the cuspidal curve can be computed by intersecting there these three classes and the class h of a hyperplane in $\mathbb{P}^3$. Now to finish the proof we use Schubert once more. $\qquad \square$

Remarks: 1) If at the end of the above proof we multiply by H instead of h, we would get the degree of the ruled surface consisting of those lines such that one of its two focal points is a cusp in the focal surface. This number turns out to be $4a + 4b + 12g - 12$, and was already known by the classics (see [18] §13 or [16] page 197). In fact, they also knew how to compute the degree invariants of the nodal and cuspidal curves. Of course, they computed all these invariants in terms of other invariants, as for instance μ_1, instead of the "modern" invariants that we use.

2) The above proposition is valid if X has not a curve of fundamental points. This hypothesis is hidden in the statement, since a fundamental point produces a singular point on the focal surface whose singularity is neither a node nor a cusp. In fact, a fundamental curve produces in the set X'' defined in the above proof a component of dimension two. However, smooth congruences with a fundamental curve are classified (see [1]).

3) The same kind of observation can be made when we have a finite number of fundamental points. The set X'' contains the cones formed by the lines of the congruence through any fundamental point. These cones do not count when intersecting with the class h, so they do not affect to the formula of $\deg C$. However, the formula in part 1) of the remark takes also account of the sum of the degrees of these cones.

Let us now apply these remarks to some examples.

Example 1.13: (See also Example 5.5 below). The complete intersection of $G(1,3)$ with a general hyperplane and a general quadric produces a congruence of bidegree $(2,2)$ and $g = 1$, in particular without cuspidal curve. As a surface in $\mathbb{P}^5$, it is the surface given by the polarized pair $(Bl_{p1,..p5}\mathbb{P}^2, 3L - E_1 - ... - E_5)$, i.e. by the linear system of plane cubics through five points. Therefore, the congruence contains sixteen lines of $\mathbb{P}^5$, which correspond to sixteen pencils of lines of $\mathbb{P}^2$. Hence the congruence contains sixteen fundamental points (and sixteen fundamental planes) and the degree of the corresponding cone at each of them is one. In fact, the formula in 1) yields 16, and hence Remark 3) proves that there are no more fundamental points (or fundamental planes).

Example 1.14: As a second example, we can consider the congruence X of bidegree $(2,3)$ which is the Del Pezzo surface (i.e. $g = 1$) given by the polarized pair $(Bl_{p1,..p4}\mathbb{P}^2, 3L - E_1 - ... - E_4)$, i.e. by the linear system of plane cubics through four points. It is known, and easy to verify, that such a Del Pezzo surface contains exactly ten lines (the four exceptional lines and the six lines joining the four base points) and five pencil of conics (the four pencils given by the lines in $\mathbb{P}^3$ through one base point and the pencil given by the conics through the base points). This implies that the congruence has 15 fundamental points and 10 fundamental planes in $\mathbb{P}^3$. Indeed the ten lines give rise to ten fundamental points and ten fundamental planes. Moreover each pencil of conics contains at least one conic which is contained in an alpha-plane. Let us prove this fact for instance for the pencil $|L - E_1|$ (for the others the proof is the same). Since the image of E_1 in $G(1,3)$ is a line, in particular it is contained in an alpha-plane, so that there is a section of $S_{|X}$ vanishing on E_1, i.e. a section of $S_{|X}(-E_1)$. Since $c_2(S_{|X}(-E_1)) = 0$, it follows easily that there is an exact sequence

$$0 \rightarrow \mathcal{O}_X(E_1) \rightarrow S_{|X} \rightarrow \mathcal{O}_X(3L - 2E_1 - E_2 - E_3 - E_4) \rightarrow 0.$$

From this it follows that $h^0(S_{|X}(-L + E_1)) = 1$, and hence any conic in $|L - E_1|$ is contained in the zero locus of a section of $S_{|X}$. But observe that $h^0(S_{|X}) = 5$, so that exactly a hyperplane inside $H^0(S_{|X})$ corresponds to alpha-planes. This means that at least one section corresponding to an alpha-plane vanishes on a a conic of the pencil, as wanted. Applying now Remark 3) we see that the degree of the ruled surface generated by the lines throught the fundamental points is 20 (since the bidegree is $(2,3)$ there is no cuspidal curve). Since we have found ten cones of degree one and five cones of degree two, there are no more fundamental points in the congruence.

Example 1.15 In this last example, we consider the congruence of bidegree $(3,3)$ and $g = 2$ which is the rational surface given by the polarized pair $(Bl_{p1,..p7}\mathbb{P}^2, 4L - 2E_1 - E_2 - ... - E_7)$, i.e. by the linear system of plane quartics with a fixed double point and through other six points. Such a Castelnuovo surface contains twelve lines (the six exceptional lines corresponding to simple points, and the six lines joining the double point with the other six ones) and 32 conics (the one corresponding to the double base point,

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the 15 corresponding to the lines joining two simple base points, the 15 corresponding to conics through the double point and other four simple points, and the one corresponding to the cubic with a double point in the double base point and passing through the other base points). Fano shows ([4], pages 154-155) that besides the twelve fundamental points coming from the twelve lines of the congruence, there can be other fundamental points (vertex of cones corresponding to conics lying in alpha-planes) or not, depending on the projective embedding. Specifically the Castelnuovo surface is the complete intersection of the cubic Segre threefold and a smooth hyperquadric in $\mathbb{P}^5$. It is hence contained in a three-dimensional linear system of hyperquadrics. Each smooth quadric in the system can be viewed as a Grassmannian. While for a general quadric we do not get extra fundamental points, for particular ones we can get one, two or three new fundamental points.

Remark: As shown in the previous example, the number of the fundamental points of a congruence does not depend only on its invariants, in particular it is meaningless to look for a formula giving the contribution of the fundamental points only in terms of the bidegree, of the sectional genus and of other usual invariants of the surface. This fact seems not to be considered by Roth who gives a formula ([16], page 198) to compute the degree ρ_2 of the scroll of lines of $\mathbb{P}^3$ consisting of those lines such that one of its two focal points is a node in the focal surface. Such a formula, when applied to a congruence of bidegree (a, b) with $a \leq 3$, hence without nodal curve, should give the number of fundamental points. However the formula for ρ_2 given by Roth fails for several congruences (and not only for the above example).

§2. Congruence of the bisecants to a space curve.

In this section we describe the congruences of the chords of a smooth irreducible skew curve Γ in $\mathbb{P}^3$.

Let Γ be a curve in $\mathbb{P}^3$ and denote by $X \subset G(1,3)$ the congruence of the bisecants to Γ. Throughout this section Γ will be assumed to be smooth, irreducible and not contained in a plane. We will also write d for the degree of Γ and p for its genus.

It is known (see [7] Theor. 2.5) that X is singular unless Γ is a rational cubic or an elliptic quartic curve. So, from now on, being mostly interested in the case of smooth congruences, we could confine ourself to consider the case of these two curves, but we prefer to study a more general situation.

Proposition 2.1. *Let Γ be as above. Then the congruence X of bisecants to Γ has bidegree $(a, b) = (\frac{1}{2}(d-1)(d-2) - p, \frac{1}{2}d(d-1))$ and sectional genus $g = \frac{1}{2}(d-2)(d-3+2p)$.*

Proof: The congruence is naturally parametrized by the second symmetric product $S = C^{(2)}$ of C. We will regard S as the quotient of $C \times C$ under the standard involution.

Let L be the line bundle giving the embedding of C into $\mathbb{P}^3$, and write L_1 and L_2 for the corresponding pullbacks of L to $C \times C$ via the two natural projections. If D denotes the diagonal of $C \times C$, there is an epimorphism $L_1 \oplus L_2 \to \mathcal{O}_D(L)$. Its kernel is invariant under the involution of $C \times C$ hence it is the pullback of a rank-two vector bundle Q on S (the so-called secant bundle). This vector bundle is the one that gives the map from S to $G(1,3)$ whose image is the congruence X.

In the intersection ring of S consider the following classes: P will represent the class of pairs containing a fixed point of C, and Δ will be the diagonal class, i.e. the image of D. We recall the following intersection numbers: $P \cdot P = P \cdot \Delta = 1$, $\Delta \cdot \Delta = 2(2 - 2g)$.

With this notation, the Chern classes of Q are $c_1(Q) = dP - \frac{1}{2}\Delta$ and $c_2(Q) = \frac{1}{2}d(d-1))$. From this one can readily obtain the bidegree by using that $a = c_1(Q)^2 - c_2(Q)$ and $b = c_2(Q)$. Notice that this bidegree could also be obtained by simple geometric arguments.

In order to obtain the sectional genus of X we need to obtain the canonical class of S. This can be easily done since $C \times C$ is a double cover of S ramified along the diagonal. We then have that numerically $K_S \equiv (2 - 2p)P + \frac{1}{2}\Delta$ and from here the wanted equality for g follows. $\qquad\square$

Remark: In the same way it is easy to find the rest of the invariants for S. In particular, $K_S^2 = 4p^2 - 13p + 9$ and $\chi(\mathcal{O}_S) = \frac{1}{2}(p-1)(p-2)$.

Definition: Let Γ be as above and consider two distinct points x, y of it. The chord $< x, y >$ through x and y is said to be *stationary* if the tangent lines t_x and t_y to Γ, at x and y respectively, are incident.

Denote by $T(x, y)$ the tangent plane to the congruence X at the point corresponding to a chord $< x, y >$. It is quite easy to verify that, if the chord $< x, y >$ is stationary, then the plane $T(x, y)$ is contained in the Grassmannian $G(1,3)$, actually it is the beta-plane generated by t_x and t_y, as we will show in the following (probably well known) lemmas.

Lemma 2.2. *Let Γ be as above and let C be the Chow complex of lines intersecting Γ. Let x be a point of Γ, consider a line L passing through x, denote by t_x the tangent line of Γ at x, and by Π the plane generated by L and t_x. Then the corresponding branch of C is smooth at the point represented by L if and only if L is different from t_x. Moreover, in this case the embedded tangent space of this branch of C at L is generated by the alpha-plane $\alpha(x)$ and the beta-plane $\beta(\Pi)$.*

Proof: This is just an easy local computation. Choose coordinates z_0, z_1, z_2, z_3 in $\mathbb{P}^3$ so that the point x becomes $(1:0:0:0)$ and the tangent line t_x is $z_2 = z_3 = 0$. Working in the open affine set $z_0 = 1$, we can parametrize Γ locally at x (which is now the origin) by $z_1 = t, z_2 = f(t), z_3 = g(t)$ with $f(0) = g(0) = f'(0) = g'(0) = 0$. Let L be the line

passing through x and through the point of coordinates $(0 : a_1 : a_2 : a_3)$, and assuming $a_1 \neq 0$, put $a_1 = 1, a_2 = u, a_3 = v$. Then a local parametrization in the open subset $\{p_{01} \neq 0\} \subset G(1,3)$ of the corresponding branch of H at the point represented by L is given by

$$(t, u, v) \mapsto (p_{02}, p_{03}, p_{12}, p_{13}) = (u, v, ut - f(t), vt - g(t))$$

Hence, the corresponding branch of H at L is smooth if and only if $(u, v) \neq (0, 0)$, i.e., if and only if L is different from the tangent line at x. In this case, the embedded tangent space of H at L has (affine) parametric equations

$$\begin{cases} p_{01} = 1 \\ p_{02} = u + \lambda \\ p_{03} = v + \mu \\ p_{12} = \nu u \\ p_{13} = \nu v \\ p_{23} = 0 \end{cases}$$

i.e. it is the projective plane $vp_{12} - up_{13} = p_{23} = 0$, which is generated by the alpha-plane $\alpha(x)$ (of equations $p_{12} = p_{13} = p_{23} = 0$) and the beta-plane $\beta(\Pi)$ (of equations $vp_{02} - up_{03} = vp_{12} - up_{13} = p_{23} = 0$).

$\square$

Lemma 2.3. *Let Γ be as above and X the congruence of bisecants to Γ. Let L be line having exactly two intersection points x and y with Γ. Then L represents a smooth point of X if and only if it is different from both t_x and t_y. In this case, denote by Π_x the plane generated by L and t_x and by Π_y the plane generated by L and t_y, then the embedded tangent space to X at L is generated by the pencils $\Omega(x, \Pi_y)$ and $\Omega(y, \Pi_x)$.*

Proof: In fact, locally at L the congruence X is the complete intersection of the two branches of the Chow complex of Γ corresponding to the points x and y. Hence, L is a smooth point of X if and only if the two branches are smooth at L and their embedded tangent spaces are different. This, due to Lemma 2.2, happens if and only if L is neither t_x nor t_y and $x \neq y$. If this is the case, the embedded tangent plane of X at L will be the intersection of the embedded tangent spaces of the two branches, which, due to Lemma 2.2, gives the thesis.

Remark: The Lemma above immediately implies that, if the chord $L = < x, y >$ is stationary, then the plane $T(x, y)$ is contained in the Grassmannian $G(1, 3)$: actually it is the beta-plane $\beta(\Pi_x) = \beta(\Pi_y)$. Since a curve has in general a one-dimensional family of stationary bisecants, the corresponding congruence of bisecants will have a focal surface, even if one would expect the focal locus to be just the curve Γ.

From Propositions 2.1 and 1.7, it follows that the degree of the (total) focal surface must be $2(d-3)(d-1+p)$, which coincides with the degree of the ruled surface of stationary bisecants to Γ (see [9], Remark 5.2). The twisted cubic is the only curve in $\mathbb{P}^3$ without stationary bisecants, so we study next in detail the only other example of smooth congruence of bisecants.

Example 2.4: Let X be the congruence of bisecants to an elliptic quartic curve $\Gamma \subset \mathbb{P}^3$. It is then known that X is a smooth congruence of bidegree $(2,6)$ and sectional genus $g = 3$. The strict focal "surface" F_0 will be Γ, while F consists of the four quadric cones containing Γ. Indeed it is easy to see that a bisecant to Γ is stationary if and only if it is contained in one of the quadric cones containing Γ. Observe that we then obtain the expected degree eight for the focal surface of X.

§3. Congruences of bitangents and flexes to a smooth surface in $\mathbb{P}^3$: global study

Let $\Sigma \subset \mathbb{P}^3$ be a surface of degree d, that we will assume, unless otherwise specified, to be smooth. In fact we will also sometimes assume Σ to be general enough, so that, for $d \geq 4$, its Picard group will be generated by the hyperplane section. Following the ideas of [12] and [21], we consider the projective bundle $p : Y = \mathbb{P}(\Omega_\Sigma(2)) \to \Sigma$. Any point of Y can be regarded as a pair (x, L), where x is a point of Σ and L is a tangent line to Σ at x. Therefore there is a map $\varphi : Y \to G(1, 3)$. In fact the twist in the projective bundle was chosen so that the tautological line bundle of Y became the pull-back of the hyperplane section of $G(1,3)$. Let us write $\mathcal{O}_Y(\ell)$ for the tautological line bundle on Y and $\mathcal{O}_Y(h)$ for the pull-back via p of the hyperplane line bundle of $\Sigma \subset \mathbb{P}^3$. In terms of vector bundles, the map φ is defined by the rank-two vector bundle Q on Y defined as a push-forward in the commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_{Y/\Sigma}(\ell - h) & \to & p^*\Omega_\Sigma(h) & \to & \mathcal{O}_Y(\ell - h) & \to & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
(3.1) \quad 0 & \to & \Omega_{Y/\Sigma}(\ell - h) & \to & p^*(P^1(\mathcal{O}_\Sigma(1))) & \to & Q & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_Y(h) & = & \mathcal{O}_Y(h) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Here the top horizontal sequence is the universal sequence on the projective bundle Y tensored with $\mathcal{O}_Y(-h)$ and the middle vertical sequence is the pull-back of the one defining

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the bundle of principal parts of $\mathcal{O}_\Sigma(1)$. The map φ is precisely defined by the composed epimorphism $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_Y \to p^*(P^1(\mathcal{O}_\Sigma(1))) \to Q$. The following closed surfaces of Y will play an important role in the sequel:

$$Y' := \{(x, L) \in Y \mid x \text{ is a parabolic point of } \Sigma\}$$

$$Y_1 := \{(x, L) \in Y \mid L \text{ is a bitangent line of } \Sigma\}$$

$$Y_2 := \{(x, L) \in Y \mid L \text{ is an inflection line of } \Sigma\}.$$

Of course, all the above sets are defined as a closure (for the definition of parabolic point, see for instance [12]).

Proposition 3.2. *The classes of Y', Y_1, Y_2 in the Picard group of Y are:* $[Y'] = 4(d-2)h$, $[Y_1] = (d+2)(d-3)\ell - 4(d-3)h$ *and* $[Y_2] = 2\ell + (d-4)h$.

Proof: The surface Y' is just the pullback via p of the parabolic curve on Σ, where it has class $4(d-2)h$, as shown in [12] (anyway, the idea is that the parabolic curve is defined by the Hessian matrix to be singular).

The class of Y_1 is computed in [21] Prop. 3.14 for $d = 4$. We essentially reproduce here Welters' ideas. Since its class is not so crutial, we chose the simplest but least general of his proofs.

If Σ is sufficiently general and $d \geq 4$, then the Picard group of Y is generated by the classes of ℓ and h. Hence the class of Y_1 will be of the form $m\ell + nh$. The first integer m is in fact the degree of the projection $Y_1 \to \Sigma$, hence it is the number of tangents at a general point of $x \in \Sigma$ that are tangent to Σ at another point.

To compute this number, consider Π the tangent plane to Σ at x and let C be the intersection of Σ with Π. Hence C is a plane curve of degree d with one ordinary node at x (hence of geometric genus $\frac{d(d-3)}{2}$) and m is the number of lines which are tangent to C outside x and pass through x. In other words, m is the number of branch points of the $(d-2) : 1$ morphism $C \to \mathbb{P}^1$ defined by the projection from x. From Hurwitz theorem one immediately gets $m = (d+2)(d-3)$.

To compute n we can use the fact that the order of the congruence of bitangents to Σ is $1/2d(d-2)(d-3)(d+3)$ (the number of bitangents of a general plane curve of degree d). Since the map $Y_1 \to G(1,3)$ (restriction of φ) is a double cover of such a congruence, it follows that $c_2(Q_{Y_1}) = d(d-2)(d-3)(d+3)$. This Chern class can be computed (with the help of the Maple package Schubert) from diagram (3.1) in terms of n, and making it equal to the second term one gets the required value of n.

The class of Y_2 can be computed in a more direct way. First we recall that inflectional tangent vectors to Σ are those in the kernel of the second fundamental form $II : Symm^2 T_\Sigma \to N_\Sigma$. Here, $N_\Sigma = \mathcal{O}_\Sigma(d)$ is the normal bundle of Σ. Hence we are

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looking at the points of Σ for which the section of $Symm^2(\Omega_\Sigma)(d)$ corresponding to II is zero. Regarding this section as a section of $Symm^2(\Omega_\Sigma(2))(d-4)$, we see from the projection formula that this corresponds to a section of $\mathcal{O}_Y(2\ell + (d-4)h)$, whose zero locus is precisely Y_2. $\qquad\square$

Let us write $X_i = \varphi(Y_i)$ and $\varphi_i = \varphi_{|Y_i}$ for $i = 1, 2$. Then X_1 is the congruence of bitangents of Σ and X_2 is the congruence of inflectional lines of Σ. They both are contained in the complex $\mathcal{H} := \varphi(Y)$ of lines tangent to Σ. What makes this approach so different among these two congruences is that, while the map $\varphi_1 : Y_1 \to X_1$ is a double cover, the map $\varphi_2 : Y_2 \to X_2$ is birational (in both cases, the map φ_i is finite as long as Σ does not contain any line). Hence we can easily compute the bidegree of both congruences, but it will be possible only for X_2 to compute all its invariants. As remarked in [21] page 30, the map φ_1 is branched over the curve of hyperflexes; the study of such a curve would certainly allow to compute all the invariants of X_1 from the ones of Y_1. We will use however a different way (see Proposition 3.5 below), which is more elegant and will also allow us to remove the genericity hypothesis for Σ.

Proposition 3.3. *The congruence X_1 has bidegree $(\frac{1}{2}d(d-1)(d-2)(d-3), \frac{1}{2}d(d-2)(d-3)(d+3))$, while the bidegree of X_2 is $(d(d-1)(d-2), 3d(d-2))$ and the sectional (geometric) genus $g = 5d^3 - 18d^2 + 14d + 1$. Moreover, the congruence X_2 is never smooth.*

Proof: The map φ_i, as a map to $G(1,3)$, is given by the rank-two vector bundle Q_{Y_i}. Since φ_1 is a double cover, then the class of X_1 is $\frac{1}{2}c_2(Q_{Y_1})$, and in fact we have already seen (or rather impose) in the proof of Prop. 3.2 that this is $\frac{1}{2}d(d-2)(d-3)(d+3)$. Analogously, its order is $\frac{1}{2}(c_1(Q_{Y_1})^2 - c_2(Q_{Y_1})) = \frac{1}{2}d(d-1)(d-2)(d-3)$, as easily computed again with the help of the Maple package Schubert.

In a similar but easier way, since φ_2 is now birational, the class of X_2 is just $b = c_2(Q_{Y_2}) = 3d(d-2)$ (which in fact corresponds to the number of flexes of a general plane curve of degree d) while its order is $a = c_1(Q_{Y_2})^2 - c_2(Q_{Y_2}) = d(d-1)(d-2)$. On the other hand, assume now that X_2 is smooth. Hence, if Σ does not contain any line, the map $\varphi_2 : Y_2 \to G(1,3)$ is necessarily an immersion, and the double-point formula for it would yield $a^2 + b^2 - c_2(N) = 0$, where N is the cokernel of the bundle inclusion $T_{Y_2} \to \varphi^* T_{G(1,3)}$. But taking into account that $\varphi^* T_{G(1,3)} \cong S_{Y_2} \otimes Q_{Y_2}$ (where S_{Y_2} is the dual of the kernel of a natural epimorphism $\mathcal{O}_{Y_2}^{\oplus 4} \to Q_{Y_2}$, and in fact the pull-back to Y_2 of the universal bundle S on $G(1,3)$), with the help once more of the Schubert package we get that the double-point formula reads

$$d(d-3)(d^4 - 3d^3 + 13d^2 - 48d + 40) = 0$$

which is absurd if $d \neq 3$. The case $d = 3$ (or more generally when Σ contains a line) is treated separately in the following lemma. All the invariants of X_2 (in particular the

sectional genus) are computed using the isomorphism with Y_2 and the fact that Y_2 is a smooth divisor of Y of a known class. ☐

Lemma 3.4. *If L is a line contained in Σ, then the corresponding point of X_2 is singular of multiplicity $3(d-2)$.*

Proof: Let us consider the point $p_L \in X_2$ corresponding to the line $L \subset \Sigma$. By abuse of notation, let us still call L to $\varphi_2^{-1}(p_L)$. In other words, we are identifying L with the curve in Y_2 which contracts to p_L. Since φ_2 is birational, the multiplicity of p_L will be precisely minus the self-intersection of L in Y_2. By adjunction we have $L^2 + K_{Y_2}L = -2$, so it is enough to prove that $K_{Y_2}L = 3d - 8$. But this is an immediate consequence of the equality $K_{Y_2} = c_1(\Omega_{Y|Y_2}) - (2\ell + (d-4)h) = (3d-8)h_{|Y_2}$, since we can then compute $K_{Y_2}L$ as the intersection in Y of $(3d-8)h$ with L. ☐

Remark: A similar statement was proved in [21] (1.1) and (1.2) for the congruence X_1 of bitangents in case $d = 4$.

Proposition 3.5. *The congruence X_1 of bitangents to a smooth surface $\Sigma \subset \mathbb{P}^3$ of degree d is smooth only for $d = 4$. The geometric genus of its hyperplane section is $g = d^5 - \frac{5}{2}d^4 - \frac{35}{2}d^3 + 60d^2 - 36d + 1$.*

Proof: The idea is to work on the Hilbert scheme $T = Hilb^2\mathbb{P}^3$ parametrizing (unordered) couples of points of $\mathbb{P}^3$ (and then study the subset of those that produce a bitangent line to Σ). Since two points (possibly infinitely close) determine a line, there is a map $q : T \to G(1,3)$. On the other hand, the set of pairs of points on a fixed line is a $\mathbb{P}^2$, parametrized by the quadratic forms (up to a constant) on the line. Therefore, the map q endows T with a projective bundle structure $T = \mathbb{P}(Symm^2Q^*)$. In this projective bundle we have the universal quadratic form given by the bundle inclusion

$$\mathcal{O}_T(-1) \hookrightarrow q^*Symm^2Q$$

which assigns at each couple of points the quadratic form (defined on the line spanned by them) vanishing on those points. We can similarly construct from this a bundle inclusion

$$\mathcal{O}_T(-2) \hookrightarrow q^*Symm^4Q$$

which corresponds for every couple to the quartic forms vanishing doubly at each of the points of the couple. The multiplication of $(d-4)$-forms by this universal form determines then another bundle inclusion i which defines the bundle R as a cokernel:

$$0 \to q^*Symm^{d-4}Q \otimes \mathcal{O}_T(-2) \xrightarrow{i} q^*Symm^dQ \to R \to 0$$

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A surface $\Sigma \subset \mathbb{P}^3$ of degree d corresponds to a section $\mathcal{O}_{G(1,3)} \to Symm^d Q$, and we are interested in the locus at which the pull-back of this section lies in the image of i. In other words, the zero locus of the corresponding section of R (obtained as the composition $\mathcal{O}_T \to q^* Symm^d Q \to R$) is the set $\tilde{X}_1$ of couples of points of Σ such that the line defined by them is tangent at those points. The congruence X_1 is the image by q of $\tilde{X}_1$. If X_1 is smooth (and Σ does not contain any line), then p defines in fact an isomorphism between $\tilde{X}_1$ and X_1, so everything reduces to computing the invariants of $\tilde{X}_1$. This is easily done by using that $\tilde{X}_1$ is defined as the zero locus of the rank-four vector bundle R, of which we can compute its Chern classes from the exact sequence defining it.

To be honest, there is a technical problem that cannot be completely solved by using the package Schubert: the Chern classes of a symmetric power of a bundle can be computed only for a fixed exponent, but not depending on a parameter d. We write the exact result we need in Lemma 3.6 below, so that the interested reader can reproduce from it all our calculations. These calculations will provide easily the sectional genus (from the product of the canonical class of $\tilde{X}_1$ and the pull-back of the hyperplane section of $G(1,3)$), as well as the rest of the invariants. In particular, one gets that, if N is the normal bundle of X_1 in $G(1,3)$, then $a^2 + b^2 - c_2(N) = \frac{1}{2}d(d-4)(d^6 - 4d^5 + 2d^4 - 20d^3 + 9d^2 + 396d - 540)$. Hence X_1 is only smooth for $d = 4$. $\qquad\square$

Lemma 3.6. *Let Q be a rank-two vector bundle on a smooth variety and let c_1, c_2 be its Chern classes. Then a symmetric power of Q has Chern classes:*

$$c_1(Symm^d Q) = \frac{1}{2}d(d+1)c_1$$

$$c_2(Symm^d Q) = \frac{1}{24}d(d-1)(d+1)(3d+2)c_1^2 + \frac{1}{6}d(d+1)(d+2)c_2$$

$$c_3(Symm^d Q) = \frac{1}{48}d^2(d-1)(d-2)(d+1)^2 c_1^3 + \frac{1}{12}d^2(d-1)(d+2)(d+1)c_1 c_2$$

$$c_4(Symm^d Q) = \frac{1}{1570}d(d-1)(d-2)(d-3)(d+1)(15d^3 + 15d^2 - 10d - 8)c_1^4$$

$$+ \frac{1}{720}d(d-1)(d-2)(d+2)(d+1)(15d^2 - 5d - 12)c_1^2 c_2$$

$$+ \frac{1}{360}d(d-1)(d-2)(d+1)(5d+12)c_2^2.$$

Proof: This is just a straightforward (but terribly annoying) calculation using the splitting principle. $\qquad\square$

§4. Congruences of bitangents and flexes to a smooth surface in $\mathbb{P}^3$: local study

In this section we analyze when a bitangent or inflectional line to a surface becomes a focal line of the corresponding congruence. We will find then that in both types of congruences we always get at least one component of the focal surface made out of focal lines. On the other hand, we will observe that the surface Σ will have a big multiplicity as a component of the focal surface. We will finally check that these two atypical situations are reflected in the formula for the degree of the focal surface, which can be derived from the invariants of the congruences computed in the previous section.

We prove first a series of local results about tangent spaces that will be useful later on.

Lemma 4.1. *Let Σ be a surface in $\mathbb{P}^3$ and let $\mathcal{H}$ be the complex of lines tangent to Σ.*
  a) *If x is a smooth point of Σ, $\Pi = T_x\Sigma$ the tangent plane of Σ at x, and L a line contained in Π passing through x, then the corresponding branch of $\mathcal{H}$ is smooth at the point represented by L if and only if the intersection multiplicity at x of L and Σ is exactly two. Moreover, in this case the embedded tangent space of this branch of $\mathcal{H}$ at L is generated by the alpha-plane $\alpha(x)$ and the beta-plane $\beta(\Pi)$.*
  b) *The surface Y_2 is singular at the points (x, L) for which x is a parabolic point (and hence L is the unique asymptotic line at x).*

Proof: This is just based on a tedious local computation to study the differential of φ at the point (x, L). Choose coordinates z_0, z_1, z_2, z_3 in $\mathbb{P}^3$ so that the point x becomes $(1 : 0 : 0 : 0)$, the plane Π has equation $z_3 = 0$ and the line L is $z_2 = z_3 = 0$. Working in the open affine set $\{z_0 = 1\}$, we can parametrize Σ locally at x (which is now the origin) by $z_3 = f(z_1, z_2)$. Hence a local parametrization of the corresponding branch of $\mathcal{H}$ at the point represented by L is given by assigning to local parameters λ, u, v the line generated by the rows of the matrix

$$(4.2) \qquad \begin{pmatrix} 1 & u & v & f \\ 0 & 1 & \lambda & f_u + \lambda f_v \end{pmatrix}$$

(f_u and f_v denoting the partial derivatives of f with respect to u and v respectively). In this way, the Plücker coordinates of this line in the open affine set of $G(1,3)$ given by $\{p_{01} = 1\}$ are:

$$p_{02} = \lambda$$
$$p_{03} = f_u + \lambda f_v$$
$$p_{12} = \lambda u - v$$
$$p_{13} = u f_u + \lambda u f_v - f$$

(These are therefore local equations for φ at (x, L)). The Jacobian matrix with respect to

λ, u, v is then

$$
\begin{pmatrix}
1 & f_u & u & u f_v \\
0 & f_{uu} + \lambda f_{uv} & \lambda & f_u + u f_{uu} + \lambda f_v + \lambda f_{uv} - f_u \\
0 & f_{uv} + \lambda f_{vv} & -1 & u f_{uv} + \lambda u f_{vv} - f_v
\end{pmatrix}
$$

We now specialize to the point represented by L (i.e. $\lambda = u = v = 0$) taking into account that $f_u(0,0) = f_v(0,0) = 0$ (since $z_3 = 0$ is the tangent plane at p) and get the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & f_{uu}(0,0) & 0 & 0 \\
0 & f_{uv}(0,0) & -1 & 0
\end{pmatrix}
$$

Hence, the corresponding branch of $\mathcal{H}$ at L is smooth if and only if $f_{uu}(0,0) \neq 0$, which is clearly equivalent to the fact that L meets Σ with multiplicity exactly two. In this case, the tangent space of $\mathcal{H}$ at L (in the embedded tangent space of $G(1,3)$ at L, which is $p_{23} = 0$) has equation $p_{13} = 0$. Hence the embedded tangent space of $\mathcal{H}$ at L is $p_{13} = p_{23} = 0$, which is generated by the alpha-plane $\alpha(x)$ (of equations $p_{12} = p_{13} = p_{23} = 0$) and the beta-plane $\beta(\Pi)$ (of equations $p_{03} = p_{13} = p_{23} = 0$). This proves a)

As for b), with the same coordinates as above, the equation of Y_2 is $f_{uu}^2 + 2 f_{uv} \lambda + f_{vv} \lambda^2 = 0$. If x is parabolic and L is the unique asymptotic line at x, then $f_{uu}(0,0) = f_{uv}(0,0) = 0$. Hence, the equation of Y_2 does not have linear monomials and therefore the point (x, L) is singular. $\qquad\square$

Lemma 4.3. *Let Σ be a surface in $\mathbb{P}^3$ and X_1 the congruence of bitangents to Σ.*
   a) *Let L be line having exactly two tangency points x and y with Σ (x and y being smooth). Then L represents a smooth point of X_1 if and only if the intersection multiplicity of L and Σ at both x and y is two. In this case, the embedded tangent space to X_1 at L is generated by the pencils $\Omega(x, T_y \Sigma)$ and $\Omega(y, T_x \Sigma)$.*
   b) *Let L be a line of X_1 having only one tangency point with Σ. Then L an Σ has intersection multiplicity at least four at the contact point. Moreover, if the intersection multiplicity is exactly four, then the line L is a smooth point of X_1 and is not contained in the focal surface.*

Proof: To prove a), we first observe that L represents to a double point of the complex of tangents $\mathcal{H}$, whose branches correspond to the image by φ of the points (x, L) and (y, L). In fact, locally at L the congruence X_1 is the complete intersection of these two branches. Hence, L will be a smooth point of X_1 if and only if the two branches are smooth at L and their embedded tangent spaces are different. This second statement is always true since $x \neq y$. Therefore, by Lemma 4.1, L is smooth if and only if the intersection multiplicity of L and Σ at both x and y is two. If this is the case, the embedded tangent plane of X_1

at L will be the intersection of the embedded tangent spaces of the two branches. Using again Lemma 4.1 and the fact that $x \neq y$, the intersection with $G(1,3)$ of the embedded tangent spaces of the two branches of $\mathcal{H}$, which are $\alpha(x) \cup \beta(T_x\Sigma)$ and $\alpha(y) \cup \beta(T_y\Sigma)$ is either $\Omega(x, T_y\Sigma) \cup \Omega(y, T_x\Sigma)$ (if $T_x\Sigma \neq T_y\Sigma$) or $\beta(T_x\Sigma)$ (if $T_x\Sigma = T_y\Sigma$). In either case, a) follows. This fact could also be deduced from the second remark after Prop. 1.7.

As for b), let L be a line of the congruence with only one tangency point x with σ. From the bundle construction in the proof of Proposition 3.5, the equation of Σ restricted to L is divisible by four times the equation of x (the universal quadratic form on L is the form vanishing twice at x, so that its square vanishes four times). Hence the intersection multiplicity of L and Σ is at least four at x.

Now we assume that L and Σ have intersection multiplicity four at x and choose coordinates as in the proof of Lemma 4.1 (so that x has affine coordinates $(0,0,0)$, L is the line $z_2 = z_3 = 0$ and the tangent plane of Σ at x is $z_3 = 0$). We can assume the local equation of Σ at x is

$$z_3 = f(z_1, z_2) = a_1 z_1 z_2 + a_2 z_2^2 + a_3 z_1^2 z_2 + a_4 z_1 z_2^2 + a_5 z_2^3$$
$$+ z_1^4 + a_6 z_1^3 z_2 + a_7 z_1^2 z_2^2 + a_8 z_1 z_2^3 + a_9 z_2^4 + \dots$$

The line of affine Plücker coordinates $p_{02}, p_{03}, p_{12}, p_{13}$ is then the one of affine equations

$$z_2 = -p_{12} + p_{02} z_1$$
$$z_3 = -p_{13} + p_{03} z_1$$

That line will be in the congruence X_1 if and only if the above substitution in the polynomial $P(z_1, z_2, z_3) = -z_3 + f(z_1, z_2)$ has two double roots. But we now observe that

$$P(z_1, p_{02} + p_{03} z_1, p_{12} + p_{13} z_1) =$$
$$= (p_{13} + a_2 p_{12}^2 - a_5 p_{12}^3 + a_9 p_{12}^4)$$
$$+ (-p_{03} - a_1 p_{12} + a_4 p_{12}^2 - 2a_2 p_{02} p_{12} + 3a_5 p_{02} p_{12}^2 - a_8 p_{12}^2 - 4a_9 p_{02} p_{12}^3) z_1$$
$$+ (a_1 p_{02} - a_3 p_{12} + a_2 p_{02}^2 + a_7 p_{12}^2 - 2a_4 p_{02} p_{12} + 6a_9 p_{02}^2 p_{12}^2 - 3a_5 p_{02}^2 p_{12} + 3a_8 p_{02} p_{12}^2) z_1^2$$
$$+ (a_3 p_{02} - a_6 p_{12} + a_4 p_{02}^2 - 2a_7 p_{02} p_{12} + a_5 p_{02}^3 - 3a_8 p_{02}^2 p_{12} - 4a_9 p_{02}^3 p_{12}) z_1^3$$
$$+ (1 + a_6 p_{02} + a_7 p_{02}^2 + a_8 p_{02}^3 + a_9 p_{02}^4) z_1^4 + \dots$$

The main point now is the technical Lemma 4.4, which we state and prove after the end of this proof. That technical lemma implies that X_1 is defined locally at L by two polynomials whose linear parts are p_{13} and $-p_{03} - a_1 p_{12}$. Hence, X_1 is smooth at L, and the embedded tangent space at that point is $p_{13} = p_{23} = p_{03} + a_1 p_{12} = 0$, which clearly is not contained in $G(1,3)$. (This tangent plane can be viewed as the only plane in the pencil determined by $\alpha(x)$ and $\beta(T_x\Sigma)$ which contains the infinitely close line to L in the quadric $z_3 = a_1 z_1 z_2 + a_2 z_2^2$, which is the osculating quadric to Σ at x). $\square$

Lemma 4.4. *Let A_d be the projective space of nonzero polynomials (up to multiplication by a nonzero constant) in $K[T]$ of degree at most d (for a fixed $d \geq 4$) and let B_d be the subset of polynomials with a factor of degree four which is a perfect square. Let the coordinates $(b_0 : \ldots : b_d)$ define the polynomial $b_0 + \ldots + b_d X^d \in A_d$. Then, locally at a polynomial $X^4 P$ (with $P(0) \neq 0$ and P square-free), B_d is defined by two affine equations in $K[b_0, \ldots, \hat{b}_4 \ldots, b_d]$ whose linear parts are b_0 and b_1.*

Proof: We start with the easy case in which $d = 4$. Then we can work in the affine space of monic polynomials, and the polynomial $b_0 + b_1 X + b_2 X^2 + b_3 X^3 + X^4$ is in B if and only if it the square of a polynomial $c_0 + c_1 X + X^2$. Therefore one gets the relations:

$$b_0 = c_0^2$$
$$b_1 = 2c_0 c_1$$
$$b_2 = 2c_0 + c_1^2$$
$$b_3 = 2c_1$$

From the last two equations one can obtain c_0 and c_1 as polynomials in b_0, b_1 without constant term, and substituting in the first two equations one gets the wanted local equations, with linear terms b_0 and b_1.

For a general d, we consider the obvious multiplication map

$$\psi : A_4 \times A_{d-4} \to A_d.$$

A polynomial as in the statement is the image of a (X^4, P), and we can assume P to have constant term equal to 1. As before, we take the obvious affine coordinates in $c_1, \ldots c_{d-4}$ in A_{d-4} and d_0, d_1, d_2, d_3 near P and X^4. Observe that X^4 becomes the origin, but P can have arbitrary coordinates $c_{10}, \ldots, c_{d-4,0}$. The map ψ is defined in these affine sets by:

$$b_0 = d_0$$
$$b_1 = c_1 d_0 + d_1$$
$$b_2 = c_2 d_0 + c_1 d_1 + d_2$$
$$b_3 = c_3 d_0 + c_2 d_1 + c_1 d_2 + d_3$$
$$b_4 = c_4 d_0 + c_3 d_1 + c_2 d_2 + c_1 d_3 + 1$$
$$b_5 = c_5 d_0 + c_4 d_1 + c_3 d_2 + c_2 d_2 + c_1$$
$$\cdots$$
$$b_{d-4} = c_{d-4} d_0 + c_{d-5} d_1 + c_{d-6} d_2 + c_{d-7} d_3 + c_{d-8}$$
$$b_{d-3} = c_{d-4} d_1 + c_{d-5} d_2 + c_{d-6} d_1 + c_{d-7}$$
$$b_{d-2} = c_{d-4} d_2 + c_{d-5} d_3 + c_{d-6}$$
$$b_{d-1} = c_{d-4} d_3 + c_{d-5}$$
$$b_d = c_{d-4}$$

We can work on the open affine $b_4 = 1$ and divide the rest of the coordinates by the above expression for b_4. It is not difficult to check that the Jacobian matrix of ψ with respect to $d_0, d_1, d_2, d_3, c_1, \ldots, c_{d-4}$ at $(0, 0, 0, 0, c_{10}, \ldots, c_{d-4,0})$ is lower triangular with 1's in the diagonal (just observe that dividing by b_4 does not change that much the aspect of the matrix). Therefore ψ is locally an isomorphism. Our hypothesis implies that (X^4, P) is the only element of $B_4 \times A_{d-4}$ whose image is $X^4 P$. We therefore get a local isomorphism between $B_4 \times A_{d-4}$ and B_d. From what we already proved for $d = 4$, the tangent space of B_4 at X^4 is given by $d_0 = d_1 = 0$. Looking at the differential of ψ we then conclude that the tangent space of B_d at $X^4 P$ is defined by $b_0 = b_1 = 0$, as wanted. $\qquad\square$

Corollary 4.5. *If $d \geq 5$, the congruence X_1 has a singular curve consisting of bitangent lines to Σ having multiplicity three at one of the tangency points. The degree of this curve in $G(1, 3)$ is $d(d - 3)(d - 4)(d^2 + 6d - 4)$.*

Proof: The first statement follows at once from Lemma 4.3. The degree of the curve can be found, for instance, in [17], art. 598 (pages 286-287). To see a modern proof, a simple way would be the following. Observe that a pair $(x, L) \in Y_1$ will belong also to Y_2 if and only if either the multiplicity of intersection of L and Σ at x is at least three (when there is another tangency point) or the intersection multiplicity is at least four (when there is only one tangency point). The second possibility produces a curve, whose degree in $G(1, 3)$ is given in Corollary 4.7 below. Once this degree is subtracted from the intersection $[Y_1][Y_2]\ell$ in Y, the remaining degree is $2d(d-3)(d-4)(d^2+6d-4)$. But, as the proof of Lemma 4.1 shows, the point of Y_2 are in the ramification locus of $\varphi : Y \to G(1, 3)$, so that the above degree is counted twice. $\qquad\square$

Lemma 4.6. *Let Σ be a surface in $\mathbb{P}^3$ and X_2 the congruence of inflectional lines to Σ.*
  a) *If L is an inflectional line to Σ at a non-parabolic point x, then L represents a smooth point of X_2 if and only if the intersection multiplicity of L and Σ at x is exactly three. In this situation, L is never contained in the focal locus of X_2, and the ramification index of $I_{X_2} \to \mathbb{P}^3$ at (x, L) is two.*
  b) *If L is an inflectional line to Σ at a parabolic point x and the intersection multiplicity of L and Σ at x is exactly three, then L represents a smooth point of X_2 and the embedded tangent plane of X_2 at L is the beta-plane $\beta(T_x\Sigma)$.*

Proof: In order to prove a), let as choose coordinates as in Lemma 4.1. Since x is not parabolic, we can also assume that the other asymptotic line of Σ at $x = (1 : 0 : 0 : 0)$ is $z_1 + z_2 = z_3 = 0$ (this apparently strange choice is made in order to guarantee that $\frac{1}{f_{vv}}$ below has a Taylor expansion). In other words, there is a local affine parametrization of Σ at x given by

$$(z_1, z_2, z_3) = (u, v, f(u, v)) = (u, v, uv + v^2 + a_0 u^3 + a_1 u^2 v + a_2 uv^2 + a_3 v^3 + \ldots)$$

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(where $+ \dots$ means that we are omitting terms of higher degree). The asymptotic lines at a point parametrized by (u, v) are given as the span of the rows of matrix (4.2), where λ is one of the roots of the equation $f_{uu} + 2f_{uv}\lambda + f_{vv}\lambda^2 = 0$. Taking into account that

$$
\begin{aligned}
f_{uu} &= 6a_0u + 2a_1v + \dots \\
f_{uv} &= 1 + 2a_1u + 2a_2v + \dots \\
f_{vv} &= 2 + 2a_2u + 6a_3v + \dots
\end{aligned}
$$

and using the Taylor expressions $\sqrt{1 + z} = 1 + \frac{1}{2}z + \dots$ and $\frac{1}{2+z} = \frac{1}{2} - \frac{1}{4}z + \dots$ to find a determination of λ in the above equation, one finds that the asymptotic lines are locally parametrized by the rows of the matrix:

$$
\begin{pmatrix}
1 & u & v & uv + v^2 + a_0u^3 + a_1u^2v + a_2uv^2 + a_3v^3 + \dots \\
0 & 1 & -3a_0u - a_1v + \dots & v + (a_1 - 6a_0)uv + (a_2 - 2a_1)v^2 + \dots
\end{pmatrix}.
$$

This gives a local affine parametrization of X_2:

$$
\begin{aligned}
p_{02} &= -3a_0u + a_1v + \dots \\
p_{03} &= v + (a_1 - 6a_0)uv + (a_2 - 2a_1)v^2 + \dots \\
p_{12} &= -v - 3a_0u^2 - a_1uv + \dots \\
p_{13} &= -v^2 - 6a_0u^3 - 6a_0u^2v - 2a_1uv^2 - a_3v^3 + \dots
\end{aligned}
$$

which must be an isomorphism at smooth points of X_2. Hence, looking at the linear part, L represents a smooth point if and only if $a_0 \neq 0$, i.e. if and only if the line L does not meet Σ with multiplicity greater than or equal to four. In this case, the embedded tangent plane is then $p_{03} + p_{12} = p_{13} = p_{23} = 0$, which is not contained in $G(1, 3)$. (This tangent plane can be interpreted as at the end of the proof of Lemma 4.3).

To compute the ramification index of $I_{X_2} \to \mathbb{P}^3$, at (x, L), just observe that the alpha plane $\alpha(x)$ is given, in the above local coordinates of $G(1, 3)$, by the equations $p_{12} = p_{13} = 0$. Look at the above value of these coordinates in the local parametrization of X_2 and using that $a_0 \neq 0$, we obtain a curvilinear scheme of degree three supported at (x, L). Therefore, the ramification index is two. This completes the proof of a).

Statement b) is proved in a similar way, but observing now that, since we are in the ramification locus of $p_{|Y_2}$, u, v is not a system of parameters for X_2 at L. Anyway, take coordinates as in Lemma 4.1 or a), and we can assume that our f takes now the form

$$
f(u, v) = v^2 + a_0u^3 + a_1u^2v + a_2uv^2 + a_3v^3 + \dots
$$

The new coordinate we have to choose now will be w, where

$$
w^2 = f_{uv}^2 - f_{uu}fvv = -12a_0u - 4a_1v + \dots
$$

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Since by hypothesis $a_0 \neq 0$, we can take v, w as a system of parameters and substitute $u = -\frac{a_1}{3a_0}v + \ldots$ in $f, f_u, f_v, f_{uu}, f_{uv}, fvv$. In particular, we get

$$\lambda = \frac{-f_{uv} + w}{f_{vv}} = \frac{a_1^2 - 3a_0a_2}{a_0}v + \frac{1}{2}w + \ldots$$

We get now a local parametrization for X_2 (substituting in (4.2)):

$$p_{02} = \frac{a_1^2 - 3a_0a_2}{a_0}v + \frac{1}{2}w + \ldots$$
$$p_{03} = \text{terms of degree } \geq 2$$
$$p_{12} = -v + \ldots$$
$$p_{13} = \text{terms of degree } \geq 2$$

This shows that L represents a smooth point of X_2 and its embedded tangent plane is $p_{03} = p_{13} = p_{23} = 0$. i.e. the beta plane $\beta(T_x\Sigma)$. □

Corollary 4.7. *If $d \geq 4$, the congruence X_2 has a singular curve consisting of the closure of non-parabolic inflectional lines meeting Σ with multiplicity at least four. The degree of this curve in $G(1,3)$ is $2d(d-3)(3d-2)$.*

Proof: The first statement is an immediate corollary of Lemma 4.6. The degree of the curve can be found in [17], art. 597 (page 286). An alternative way of computing this degree is to use the construction in the proof of 3.5. The universal quadratic form can be also viewed as a map $q^*Q^*(-1) \to Q$, so that its determinant (whose zeros correspond to the pairs of coincident points) is a section of $(\bigwedge^2 Q)^{\otimes 2}(2)$. Intersecting $\tilde{X}_1$ with that class and the class of a hyperplane one gets the wanted number. Of course, a better way would be to work directly on $\mathbb{P}(Symm^4Q^*)$. □

Remarks: 1) From the invariants of the congruence of bitangents X_1 found in Props. 3.3 and 3.5, the degree of the (total) focal surface F of X_1 must be $d(d-3)(2d^3+2d^2-35d+26)$. Clearly, the strict focal surface F_0 is Σ. As already noticed in the proof of Prop. 3.2, the map $Y_1 \to \Sigma$ has degree $(d+2)(d-3)$, so that F_0 counts with multiplicity $(d+2)(d-3)$ in F. Therefore, F has still some extra components of total degree $2d(d-3)(d^3 + d^2 - 18d + 12)$.

2) Similarly, from the invariants of the congruence X_2 of flexes to Σ found in Prop. 3.3, the degree of the total focal surface F of X_2 is $2d(6d^2 - 21d + 16)$. The strict total surface is again Σ. Since through a general point of Σ there are two asymptotic lines and the ramification at each of them is two (see Lemma 4.6), Σ now counts with multiplicity four. Hence, the extra components of F have total degree $2d(6d^2 - 21d + 14)$.

The following propositions will explain where these extra components come from.

Proposition 4.8. *Let Σ be a general surface $\Sigma \subset \mathbb{P}^3$ of degree d and let $X_1 \subset G(1,3)$ be the congruence of bitangents to Σ. Then there are two curves of X_1 all of whose lines are entirely contained in the (total) focal surface F of X_1: The singular curve of Corollary 4.5 and the curve of stationary bitangents to Σ (i.e. bitangents such that the tangent plane to Σ at the two tangency points is the same). Moreover, the degree of the ruled surface consisting of such stationary bitangents has degree $d(d-2)(d-3)(d^2+2d-4)$.*

Proof: Let L be a bitangent tangent to Σ. If there is only one tangency point, by Lemma 4.3then L has intersection multiplicity at least four at the contact point and, if this multiplicity is exactly four, then L not contained in the focal locus. But, if Σ is general, the set of lines with intersection multiplicity at least five at some point of Σ should be finite (and there would be precisely $5d(d-4)(7d-12)$ such lines). Hence there is no curve of focal lines whose general element is tangent at two infinitely close points.

Assume now that that there are two different tangency points x_1, x_2. Suppose first that L has intersection multiplicity at least three at some of the points. Then, by Lemma 4.3, L is a cuspidal point of X_1. Therefore, for any point $x \in L$, the line counts at least twice as a line of the congruence passing through x, which means that L is entirely contained in the focal locus.

So we assume that L is simply tangent at x_1 and x_2, and let Π_1, Π_2 be the respective embedded tangent planes to Σ. Obviously the line L (and hence also the points x_1 and x_2) is contained in both Π_1 and Π_2. Then, by Lemma 4.3, the tangent plane to X_1 (as a surface in $\mathbb{P}^5$) at the point represented by L is generated by the pencils $\Omega(x_1, \Pi_2)$ and $\Omega(x_2, \Pi_1)$. Therefore, it is clear that this plane is contained in $G(1,3)$ (and is in fact a beta-plane) if and only if $\Pi_1 = \Pi_2$.

Finally, the degree of the ruled surface of stationary bitangents can be found in [17], art. 613 (page 305). $\qquad\blacksquare$

Remark: Observe that X_1 possesses another singular curve, namely the curve of tritangent lines. This is a curve of degree $\frac{1}{3}d(d-3)(d-4)(d-5)(d^2+3d-2)$, from [17], art. 599, pages 287-288. However, it is a triple nodal curve (while the curve of Corollary 4.5 is a cuspidal curve). This is what makes that its lines are not properly focal lines.

Proposition 4.9. *Let Σ be a general surface $\Sigma \subset \mathbb{P}^3$ of degree d and let X_2 be the congruences of flexes to Σ. Then there are two curves of X_2 all of whose lines are entirely contained in the (total) focal surface F of X_2: The singular curve of Corollary 4.7 and the curve of parabolic inflectional lines to Σ. Moreover, the degree of the ruled surface of parabolic inflectional lines to Σ has degree $2d(d-2)(3d-4)$.*

Proof: By Lemma 4.6, a curve consisting of focal lines such that its general element is non-parabolic must be the singular curve of asymptotic lines with intersection multiplicity

at least four. As in the previous Proposition 4.8, that curve clearly consists of focal lines.

Assume now that a general line of such a curve is parabolic. By Lemma 4.6, a general point of such a curve (i.e. a line having intersection multiplicity three at the tangency point) is a focal line. The degree of the ruled surface of asymptotic lines at parabolic points can be obtained as follows (of course, it can also be found in [17], art. 576, Ex. 3, page 255):

We observe from Lemma 4.1 that the surface Y_2 is double along its intersection with the surface Y' (of pairs $(x, L) \in Y$ with x parabolic). Therefore, the degree of their set-theoretical intersection will be $\frac{1}{2}[Y_2][Y']\ell$. From Prop. 3.2, an easy calculation shows that the wanted degree is $2d(d-4)(3d-4)$. □

§5. Smooth congruences of bitangents to arbitrary surfaces in $\mathbb{P}^3$.

In the previous section we dealt with congruences of bitangents and flexes to smooth surfaces and, with the only exception of the bitangents to a smooth quartic surface, we always got singular congruences. However, our scope is to find smooth congruences. On the other hand, we have seen that all lines of a smooth congruence are bitangent to their focal surface, which is in general very singular. So it is natural to study congruences of bitangents to arbitrary surfaces in $\mathbb{P}^3$, hoping to then understand any smooth congruence. The main problem is then how to compute the invariants of of such a congruence. The bidegree is not difficult to find. We will give it when the singularitites of Σ and Σ^* are not too bad:

Lemma 5.1. *Let $\Sigma \subset \mathbb{P}^3$ be a surface of degree d, class d^*, class of the hyperplane section μ_1, ordinary nodal curve of degree δ, ordinary cuspidal curve of degree κ and no other singular curves. Assume the same hypothesis for the singular locus of the dual surface holds, and let δ^* be the number of bitangent planes through a point and κ^* the number of inflectional planes through a point. Then the bidegree of the congruence X of bitangents to Σ is (a, b) with $a = \frac{1}{2}(\mu_1^2 - 3\kappa^*) + 4d^* - 5\mu_1$ and $b = \frac{1}{2}(\mu_1^2 - 3\kappa) + 4d - 5\mu_1$.*

Proof: The class b is the number of lines of X in a general plane of $\mathbb{P}^3$, i.e. the number of bitangents of a general hyperplane section of Σ. This hyperplane section has degree d, class μ_1, δ nodes and κ cusps. Then, from Plücker formulas (see for instance [20], V§8.2) we get that $d = \mu_1(\mu_1 - 1) - 2b - 3i$, $\kappa = 3\mu_1(\mu_1 - 2) - 6b - 8i$ (where i is the number of flexes of the curve). From this we immediately get the wanted value for b. The value of a is obtained by duality. □

We start now a series of examples to try to illustrate what the general situation should be.

Example 5.2: The hypothesis on the dual of Σ is really needed. For instance, consider the tangent developable of a twisted cubic C. This is a a quartic surface Σ whose singular locus is C, which appears as a cuspidal locus. Hence, $d = 4$, $\mu_1 = 3$ (its hyperplane section is a rational quartic with three cusps, so its dual is a nodal cubic), $\delta = 0$ and $\kappa = 3$. Then we get $b = 1$ (in fact, as we remarked, the dual of the hyperplane section of Σ has one node). But $\kappa^* = \delta^* = d^* = 0$, since Σ^* is a curve. Then the formula for a is not valid (fortunately, because the corresponding value would be $a = -\frac{21}{2}$, negative and not an integer!). The correct value can be computed as follows.

Let L be a bitangent line with tangency points x_1 and x_2. Then obviously L is the intersection of the tangent planes $T_{x_1}\Sigma$ and $T_{x_2}\Sigma$. But the converse is also true. Take two planes Π_1, Π_2 tangent to Σ. Since Σ is developable, they are tangent respectively along lines L_1, L_2. Let L be the intersection of Π_1 and Π_2. Then L meets L_1 in a point x_1 and meets L_2 in a point x_2. It is now clear that L is a bitangent line with contact points x_1 and x_2. With this description, the dual congruence will be the congruence of bisecants to the dual Σ^* (which is a twisted cubic). This dual congruence has bidegree $(1, 3)$, so that our congruence has bidegree $(3, 1)$. Its total focal surface has degree four (and a cuspidal curve), so it is precisely Σ (contrary to the situation for a smooth surface in $\mathbb{P}^3$, as we have seen in Prop. 4.8). Hence the congruence is the set of bitangents to its focal surface (total or strict). This is not going to be however the situation for a "general" congruence.

Example 5.3: The above example shows that the dual of the congruence of bisecants to a twisted cubic behaves nicely with respect to is focal surface. So it is natural to see what happens to the congruence X dual of the other smooth congruence of bisecants, namely the bisecants to an elliptic quartic C. Then X has bidegree $(a, b) = (6, 2)$ and sectional genus $g = 3$. Hence, the total focal surface has degree 16. On the other hand, reasoning as in the previous example, X will be the congruence of bitangents to the dual C^*, which is a tangent developable of degree 8 and cuspidal curve of degree 12 (corresponding to the osculating planes of C^*). Since the hyperplane section has genus one, it follows easily that Σ has a nodal curve of degree $\delta = 8$ and hence $\mu_1 = 4$. What happens now is that the total focal surface is twice $\Sigma = C^*$ (and therefore no formula for the invariants of the focal surface is valid anymore). Indeed, given a general point $x \in \Sigma$, there are two bitangents to Σ with tangency points at x and another point. Summing up, the congruence of bitangents to the (strict) focal surface coincide with the congruence X itself, but the (total) focal surface of the congruence is not Σ as a scheme, but only as a set.

Example 5.4: We have observed (Prop. 3.5 or Corollary 4.5) that the only smooth congruence of bitangents to a smooth surface in $\mathbb{P}^3$ is the congruence of bidegree $(12, 28)$ of bitangents to a smooth quartic $\Sigma \subset \mathbb{P}^3$. By duality, we also have a smooth congruence X of bidegree $(28, 12)$ consisting of the bitangents to the dual Σ^*. This is a surface in $\mathbb{P}^3$ of

degree 36, a nodal curve of degree 480 and cuspidal curve of degree 96. As in the dual case, this counts six times in the total focal surface (since through a general point of it there pass six lines that are tangent at that point and another one). But the total focal surface has degree 16, so that there are no other components. Hence this congruence verifies the same property with respect to the focal surface as the one in the previous example.

Example 5.5: Consider in $G(1,3)$ the congruence X obtained in Example1.13, which has bidegree $(2,2)$ and sectional genus $g = 1$. It is then a very classical result that the focal surface is the so-called Kummer's surface, a quartic surface with sixteen nodes, corresponding to the sixteen fundamental points of X (see for [8] for a thorough study of this surface). However, the congruence of bisecants to the Kummer's surface (which should have bidegree $(12, 28)$) splits as sixteen beta-planes (corresponding to the singular planes) and six congruences of bidegree $(2, 2)$ as above.

We conjecture that the general situation should be like the above example (except for the existence of fundamental points). In other words, a "general congruence" should have an irreducible reduced focal surface (i.e. the total focal surface coincides with the strict focal surface), and the congruence of bitangents to the focal surface splits as the original congruence plus another congruence (in general irreducible). Observe that the fact that the congruence of bitangents to F splits implies that one does not need to expect to have excedentary components for the focal surface (as it should happen for the congruence of all bitangents to a surface, as remarked in Prop. 4.8). Now we explicitly state our conjectures:

Conjecture 5.6. *If the total focal surface of a smooth congruence X is not irreducible, then either X is the congruence of secant lines to a curve in $\mathbb{P}^3$ (hence necessarily the one in Example 2.4), or a congruence of bitangents to a surface in $\mathbb{P}^3$ or a congruence of flexes to a surface in $\mathbb{P}^3$.*

Conjecture 5.7. *It the total focal surface of a smooth congruence X is not reduced, the either X is the congruence of bitangents to a surface in $\mathbb{P}^3$ or a congruence of flexes to a surface in $\mathbb{P}^3$.*

These conjectures can be strengthen with the three following ones:

Conjecture 5.8. *If the congruence of bitangents to a surface $\Sigma \subset \mathbb{P}^3$ is smooth, then either Σ is a smooth quartic surface, or its dual (see Example 5.4) or the tangent developable of a twisted cubic (see Example 5.2) or the one in Example 5.3.*

Conjecture 5.9. *There is no smooth congruence of flexes to any surface in $\mathbb{P}^3$. More generally, there are no congruences of the third class of Goldstein classification.*

Conjecture 5.10. *Let X be a smooth congruence and let F_0 be its strict focal surface. Then X coincides with the congruence of bitangents to F_0 only in the case of Examples 5.2, 5.3, 5.4or its dual $(12, 28)$ of bitangents to a smooth quartic surface.*

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